

# Classical approximation of the Boltzmann equation in high energy QCD

*T. Stockamp*<sup>1</sup>

Fakultät für Physik, Universität Bielefeld,  
D-33501 Bielefeld, Germany

## Abstract

Recently, Mueller and Son discussed the time evolution of a dense system towards equilibrium in a scalar  $\lambda\varphi^4$  field theory [1]. They show the equivalence of the classical field approximation and the Boltzmann equation in all but linear terms in the occupation number. Here we present the generalization to high energy QCD.

<sup>1</sup>*mail* : stockamp@physik.uni-bielefeld.de

# 1 Introduction

In high energy heavy ion collision experiments it is believed that a new state of matter, the so-called quark gluon plasma, may be created ([2], [3]). Theoretically it is therefore important to understand the time evolution of dense strongly interacting matter towards thermal equilibrium. This evolution can be described in terms of the gluon occupation number  $f$ . In the very early stages after a collision  $f$  is of the order  $\alpha_s^{-1}$  ([4], [5]) where  $\alpha_s$  is the strong coupling constant. As the system evolves  $f$  decreases and is finally of the order 1 at equilibrium.

Unfortunately there is no general theoretical framework to describe the complete evolution. For large  $f$  (i.e. very early times) non-linear classical field theory can be applied while a Boltzmann equation that takes quantum fluctuations into account (this will be called full or “quantum“ Boltzmann equation) may be used near equilibrium.

In order to obtain a continuous description of the evolution one may try to fit the transition between the two approaches by a classical field approximation of the full Boltzmann equation (Fig. 1). However, it is a priori not

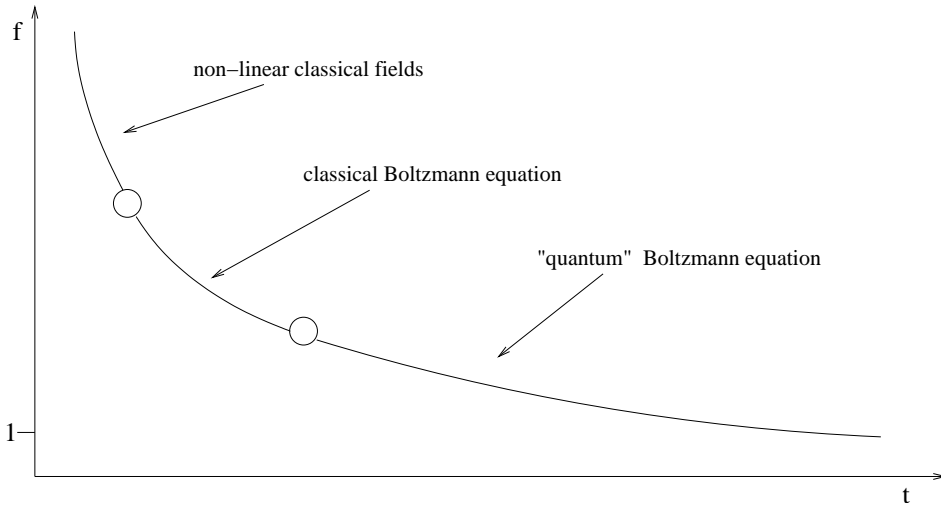


Figure 1: time evolution of the gluon occupation number

clear that these transitions are as smooth as the above figure suggests. In this paper we compare the classical and quantum Boltzmann equations in high energy QCD and show that they are equivalent to orders  $f^3$  and  $f^2$ . The comparison of non-linear classical fields with the classical Boltzmann equation is an entirely different problem and will not be treated here.

The structure of the paper is as follows: in section 2 we briefly review the main results of Mueller and Son [1] where the basis for our considerations is developed within a scalar field theory. In section 3 we generalize their strategy to high energy QCD and show that one obtains essentially the same results.

We want to emphasize that our argumentations are mostly qualitative. Perhaps surprisingly, we need very few computations to derive our results; some simple topological and diagrammatical considerations are sufficient. Similar methods can be found, e.g., in [6].

## 2 Scalar field theory

### 2.1 Separation of classical and quantum contributions

Our starting point is the scalar field theory with  $\lambda\varphi^4$  interaction given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4. \quad (1)$$

Throughout this paper we assume coupling constants to be small and perform the relevant computations in first order perturbation theory.

Our ultimate interest will be to gain deeper insight into the time evolution of heavy ion collisions. So we have to generalize our theory to finite temperature. It is convenient to apply the closed time path formalism (CTP) which leads to a doubling of the field variables ([7], [8]):  $\varphi \rightarrow \Phi_-, \Phi_+$ .

The Lagrangian then reads

$$\begin{aligned} \mathcal{L}_{CTP} = & \frac{1}{2}\partial_\mu\Phi_-\partial^\mu\Phi_- - \frac{1}{2}m^2\Phi_-^2 - \frac{\lambda}{4!}\Phi_-^4 \\ & - \left( \frac{1}{2}\partial_\mu\Phi_+\partial^\mu\Phi_+ - \frac{1}{2}m^2\Phi_+^2 - \frac{\lambda}{4!}\Phi_+^4 \right). \end{aligned} \quad (2)$$

We now perform a change of the field variables in order to distinguish the classical field (denoted  $\Phi$ ) and quantum fluctuations (denoted  $\Pi$ )

$$\Phi = \frac{1}{2}(\Phi_- + \Phi_+) \quad \Pi = \Phi_- - \Phi_+ \quad , \quad (3)$$

$$\Phi_- = \Phi + \frac{1}{2}\Pi \quad \Phi_+ = \Phi - \frac{1}{2}\Pi \quad . \quad (4)$$

As we are interested in systems where both  $\Phi_+$  and  $\Phi_-$  are large, the above interpretations of  $\Phi$  and  $\Pi$  are already at least qualitatively justified.

In terms of these new fields the Lagrangian becomes

$$\mathcal{L}_{\Phi\Pi} = \partial_\mu \Phi \partial^\mu \Pi - m^2 \Phi \Pi - \frac{\lambda}{3!}(\Phi^3 \Pi + \frac{1}{4} \Phi \Pi^3) \quad . \quad (5)$$

Indeed, it can be easily shown now by neglecting higher than linear terms in  $\Pi$  that  $\Phi$  fulfills the classical equation of motion (for details see [1])

$$(\square + m^2)\Phi + \frac{\lambda}{3!}\Phi = 0 \quad . \quad (6)$$

From the Lagrangian (5) one can identify the vertices of the theory (Fig. 2).

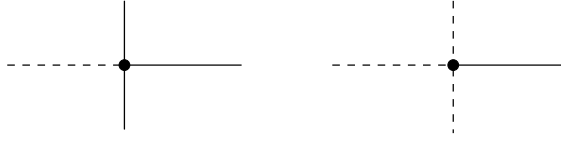


Figure 2: vertices of the scalar theory

The full lines correspond to the classical field  $\Phi$  and the dashed lines to the quantum fluctuations  $\Pi$ . The diagram on the left will be called classical vertex, the right one quantum vertex.

The most important quantities for our argumentations will be the free Greens functions or propagators for the  $\Phi$  and  $\Pi$  fields, respectively. In a more rigorous discussion one should adopt the full propagators, but mass corrections are negligible as long as the coupling constant is small enough, i.e.  $f\lambda \ll 1$ . (For details on this point see [1].)

Note that there are also mixed propagators describing the change from  $\Phi$  to  $\Pi$  and vice versa. One obtains

$$G_{\Phi\Phi} = 2\pi\delta(p^2 - m^2)(f + \frac{1}{2}) \quad , \quad (7)$$

$$G_{\Pi\Phi} = \frac{i}{p^2 - m^2 - i\varepsilon p_0} \quad , \quad (8)$$

$$G_{\Phi\Pi} = \frac{i}{p^2 - m^2 + i\varepsilon p_0} \quad , \quad (9)$$

$$G_{\Pi\Pi} = 0 \quad . \quad (10)$$

The crucial observation here is that only  $G_{\Phi\Phi}$  depends on the occupation number  $f$ .

## 2.2 Boltzmann equation

The Boltzmann equation describes the time evolution of the occupation number  $f$  of a given state as the difference between the scattering of particles into and out of this state (gain and loss). These scatterings are included in the collision term  $C$ . Considering the scalar field theory (1) one may express the collision term to lowest order diagrammatically (Fig. 3).

$$C(p) = \left| \begin{array}{c} \text{diagram: gain} \end{array} \right|^2 - \left| \begin{array}{c} \text{diagram: loss} \end{array} \right|^2$$

"gain"                      "loss"

Figure 3: collision term to lowest order

For a detailed analysis of the Boltzmann equation in scalar and gauge field theories see [9]. For our purposes it is sufficient to examine the topology of the involved diagrams and it is easy to see that these are effectively sunset graphs (Fig. 4).

$$\left| \begin{array}{c} \text{diagram: sunset} \end{array} \right|^2 = \text{diagram: sunset topology}$$

Figure 4: collision term topology

When we express now our theory in terms of  $\Phi$  and  $\Pi$  we use the corresponding vertices. The classical approximation then consists in retaining only collision term diagrams without quantum vertices.

There are three different classical diagrams (Fig. 5).

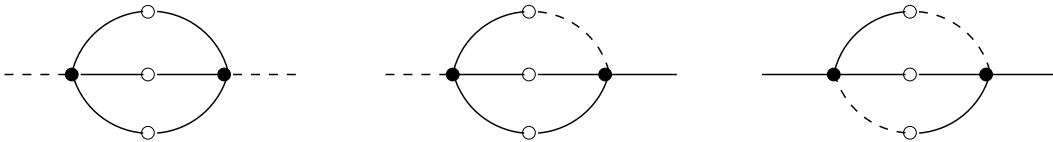


Figure 5: classical contribution to  $C(p)$

In Fig. 5 the left diagram is proportional to  $(f + \frac{1}{2})^3$  as three propagators  $G_{\Phi\Phi}$  appear. (Here and in the following empty circles appear in propagators while

full circles stand for vertices.) Similarly, the other graphs are proportional to  $(f + \frac{1}{2})^2$  and  $(f + \frac{1}{2})^1$ , respectively.

In the full Boltzmann equation one also has to include the diagrams with one quantum vertex (Fig. 6). In Fig.6 the left diagram is proportional to

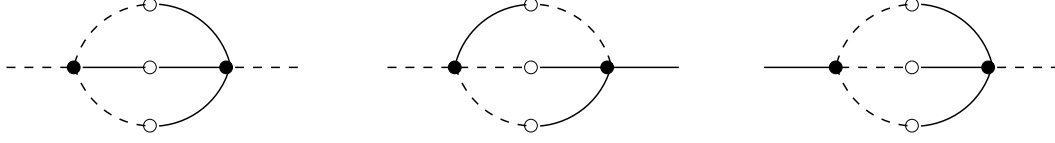


Figure 6: quantum contribution to  $C(p)$

$(f + \frac{1}{2})^1$  while the others are independent of  $f$ .

There are no first order diagrams with two quantum vertices as at least one propagator  $G_{III} = 0$  would appear. So we can conclude that classical field theory and the Boltzmann equation are equivalent in orders  $f^3$  and  $f^2$  ([1]). In the next section we will show how this strategy can be generalized to QCD of heavy ion collisions.

### 3 High energy QCD

#### 3.1 Separation of classical and quantum contributions

The results of the previous section may be generalized to QCD by making two approximations which are well established in the context of heavy ion collisions.

First, we will work within the gluon saturation scenario ([10], [11], [12], [13]). This means that we may neglect the fermionic degrees of freedom in our system. So the QCD Lagrangian simplifies considerably and reads in the CTP formalism

$$\mathcal{L}_{CTP} = -\frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}[A_{a\mu}^-] + \frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}[A_{a\mu}^+] \quad . \quad (11)$$

Here the letter ‘a’ is a color index and  $A_\mu$  the gluon field. In analogy to the field transformation in the scalar case we now define

$$\Phi_{a\mu} = \frac{1}{2}(A_{a\mu}^- + A_{a\mu}^+) \quad \Pi_{a\mu} = A_{a\mu}^- - A_{a\mu}^+ \quad , \quad (12)$$

$$A_{a\mu}^- = \Phi_{a\mu} + \frac{1}{2}\Pi_{a\mu} \quad A_{a\mu}^+ = \Phi_{a\mu} - \frac{1}{2}\Pi_{a\mu} \quad . \quad (13)$$

This leads to the following expression for  $F_{\mu\nu}^{a+}$

$$\begin{aligned}
F_{\mu\nu}^{a+} &= \partial_\mu A_\nu^{a+} - \partial_\nu A_\mu^{a+} + g f^{abc} A_\mu^{b+} A_\nu^{c+} \\
&= \partial_\mu \Phi_\nu^a - \partial_\nu \Phi_\mu^a + g f^{abc} \Phi_\mu^b \Phi_\nu^c \\
&\quad - \frac{1}{2} (\partial_\mu \Pi_\nu^a - \partial_\nu \Pi_\mu^a - \frac{1}{2} g f^{abc} \Pi_\mu^b \Pi_\nu^c) \\
&\quad - \frac{1}{2} g f^{abc} (\Phi_\mu^b \Pi_\nu^c + \Pi_\mu^b \Phi_\nu^c) \quad ,
\end{aligned} \tag{14}$$

and similarly for  $F_{\mu\nu}^{a-}$ .

Next we express the Lagrangian (11) in terms of the fields  $\Phi_\mu$  and  $\Pi_\mu$ . Neglecting higher than linear terms in the quantum fluctuations  $\Pi_\mu$  one obtains

$$\mathcal{L}_{\text{linear}} = (D_\mu^{ab} F_b^{\mu\nu}[\Phi]) \Pi_\nu^a \tag{15}$$

with

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abc} \Phi_\mu^c \quad , \tag{16}$$

$$F_b^{\mu\nu}[\Phi] \equiv \partial^\mu \Phi_b^\nu - \partial^\nu \Phi_b^\mu + g f^{bcd} \Phi_c^\mu \Phi_d^\nu \quad . \tag{17}$$

Thus,  $\Phi_\mu$  fulfills the classical equation of motion

$$D_\mu^{ab} F_b^{\mu\nu}[\Phi] = 0 \quad . \tag{18}$$

So our interpretation of  $\Phi_\mu$  as classical field is justified in complete analogy to the scalar case.

From the equations (15) to (17) one also sees that there is only one classically allowed vertex in first order in the coupling constant  $g$  (Fig. 7).

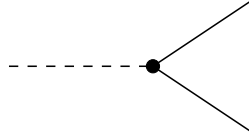


Figure 7: classical vertex in QCD

Here full lines denote classical fields and dashed lines quantum corrections like in the previous section.

In order to obtain the quantum couplings one has to take into account the terms nonlinear in  $\Pi_\mu$ . The corresponding Lagrangian may be easily computed and reads

$$\mathcal{L}_{nonlin} = -\frac{1}{4}gf_{abc}\Pi^{b\mu}\Pi^{c\nu}\left\{\frac{1}{2}(\partial_\mu\Pi_\nu^a - \partial_\nu\Pi_\mu^a) + gf^{ade}\Phi_\mu^d\Pi_\nu^e\right\} \quad . \quad (19)$$

So we have one first order quantum coupling (Fig. 8).

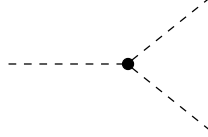


Figure 8: quantum vertex in QCD

### 3.2 The collision term

Our second approximation leads to a simple topology of the collision term C. As fermions are neglected the only contributions to C come from gluon scattering. We now assume that the t-channel dominates, where t is the Mandelstam variable (see, e.g., [14]). (In fact, our only assumption is to work in the high energy limit where both approximations are valid.)

The t-channel gluon scattering is shown diagrammatically in Fig. 9.



Figure 9: t-channel gluon scattering

This makes the relevant collision term topology quite simple as can be seen in Fig. 10 (note that in contrast to the previous section only three-field vertices are present).

As the main features of the propagators do not change, especially  $G_{\Phi\Phi} \propto (f + \frac{1}{2})$  and  $G_{\Pi\Pi} = 0$ , we are now again ready to compare classical field theory with the full Boltzmann equation. As before the classical approximation consists in neglecting diagrams with quantum vertices which are only taken into account in the full collision term.



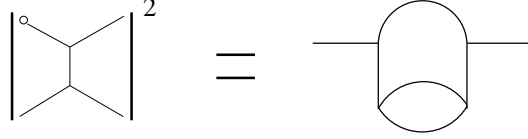


Figure 10: collision term topology in QCD

Let us consider for example the diagram in Fig. 11 that clearly has the required topology.

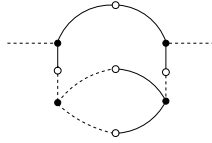


Figure 11: quantum contribution to  $C(p)$

It contains one quantum vertex so it is not included in the classical approximation. The contribution from this graph to the collision term is proportional to  $(f + \frac{1}{2})$  as one propagator  $G_{\Phi\Phi}$  appears. Similarly, the diagram in Fig. 12 is classical and proportional to  $(f + \frac{1}{2})^3$ . It is easy to check that these examples

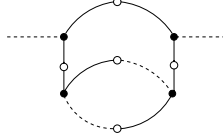


Figure 12: classical contribution to  $C(p)$

give already the highest order contributions to the classical approximation and the quantum corrections, respectively.

We conclude that the results of the scalar case are indeed valid in high energy QCD: The classical field approximation and the quantum Boltzmann equation match in orders  $f^3$  and  $f^2$ , i.e. in leading orders of the gluon occupation number.

## 4 Acknowledgments

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